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On the convergence domains of the p-cyclic SOR

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Abstract

For the solution of the linear system $Ax = b$, where A is block *p-cyclic*, the block SOR iterative method is to be considered. Suppose that the block Jacobi iteration matrix B, associated with A, has eigenvalues whose pth powers are all real of the same sign. The problem of the determination of the precise convergence domains of the SOR method in case A is also *consistently ordered* was solved by Hadjidimos, Li and Varga by using the Schur–Cohn algorithm. The same convergence domains were later recovered by other approaches too; specifically, Wild and Niethammer and also Noutsos, independently, used hypocycloidal curves. In this manuscript it is assumed that A is *not* consistently ordered but A^{T} is. By using the Schur–Cohn algorithm we successfully determine, not only: (i) the precise SOR convergence domains, but also (ii) intervals for $\rho(B)$, the spectral radius of B, that directly imply that the optimal value of the SOR relaxation factor ω is equal to 1. In this work new results are obtained, some well-known ones are recovered or confirmed and a number of theoretical examples are investigated further. It is worth noting that among the new results, we derived something *not* quite expected; specifically, in many cases there exist pairs $(\rho(B), \omega)$ for which the SOR method associated with the matrix A we consider converges while the corresponding SOR for the p-cyclic consistently ordered matrix A^{T} does *not*!

Keywords: Iterative method; Successive overrelaxation (SOR) method; p-cyclic matrix; Schur-Cohn algorithm

AMS classifications: Primary 65F10, CR categories 5.14

1. Introduction and preliminaries

For the solution of the nonsingular linear system

$$Ax = b, \tag{1.1}$$

where $A \in \mathbb{C}^{n,n}$ and x, $b \in \mathbb{C}^n$, the block successive overrelaxation (SOR) method is considered. Suppose that A is partitioned in the $p \times p$ block form

$$A = D(I - L - U), \tag{1.2}$$

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where D is a $p \times p$ block diagonal nonsingular matrix and L and U are block strictly lower and strictly upper triangular matrices, respectively. As is known for the solution of the system (1.1)–(1.2) the block SOR method is defined by

$$x^{(m+1)} = \mathcal{L}_\omega x^{(m)} + \omega(I - \omega L)^{-1}D^{-1}b, \quad m = 0,1,2,\ldots,$$
$$\mathcal{L}_\omega := (I - \omega L)^{-1}[(1-\omega)I + \omega U]. \tag{1.3}$$

In (1.3), $x^{(m)}$ is the mth approximation to the solution of (1.1), with $x^{(0)} \in \mathbb{C}^n$ arbitrary, $\omega \neq 0$ the relaxation factor and $\mathcal{L}_\omega$ the SOR iteration matrix. From Kahan's work [6] it is known that a *necessary* condition for (1.3) to converge to the solution of (1.1)–(1.2) is $|\omega - 1| < 1$ which, if we restrict to real values of ω, is equivalent to $0 < \omega < 2$. Moreover, a *necessary* and *sufficient* condition for (1.3) to converge is $\rho(\mathcal{L}_\omega) < 1$, with $\rho(\cdot)$ denoting spectral radius (see [1, 14], or [18]).

For the study of the convergence properties of the SOR method (1.3), one usually considers the block Jacobi iteration matrix associated with A in (1.2), namely

$$B := L + U. \tag{1.4}$$

This is because information about the spectrum of B, denoted by $\sigma(B)$, is necessary in order to enable one to answer the following two questions:

(i) for what pairs $(\rho(B), \omega)$ does (1.3) converge? and

(ii) for a given $\rho(B)$, for which convergence of (1.3) is guaranteed, what is the (optimal) value of ω that minimizes $\rho(\mathcal{L}_\omega)$ and makes therefore (1.3) converge (asymptotically) in the fastest possible way?

Complete answers to questions (i) and (ii) above have only been given for particular classes of matrices A in case certain information regarding $\sigma(B)$ is available. For example, many results have been obtained in the case where A belongs to the class of block p-cyclic consistently ordered matrices (cf. [14]) or, more generally, to the class of block generalized consistently ordered $(p-q,q)$-matrices (or $(p-q,q)$-GCO matrices) (cf. [18]). It is noted that the former class of consistently ordered matrices is a subclass of the latter one corresponding to $q = p-1$.

In case A belongs to the class of p-cyclic matrices the analysis and study of the SOR convergence may be accomplished. This is mainly due to the fact that the sets of eigenvalues $\mu \in \sigma(B)$ and $\lambda \in \sigma(\mathcal{L}_\omega)$ are connected by means of the functional equation

$$(\lambda + \omega - 1)^p = \mu^p \omega^p \lambda^q \tag{1.5}$$

first given by Varga [14] and then by Verner and Bernal [15]. Eq. (1.5) generalizes the famous equations by Young and Varga which correspond to $(p,q) = (2,1)$ and $(p,q) = (p,p-1)$, respectively.

The analysis of the SOR convergence is facilitated further if one assumes that besides A being p-cyclic the eigenvalue spectra of B^p are real of the same sign. The reader is referred to some of the basic works in which optimal values for the parameter ω were determined when $\sigma(B^p)$ is nonnegative (e.g., [8, 13, 17]), as well as when $\sigma(B^p)$ is nonpositive (e.g., [2, 3, 7, 10, 9, 16]). In all of the works just mentioned, except [8, 3], A is assumed to be a p-cyclic consistently ordered matrix. The very first works concerned with the determination of the convergence domains of the SOR method were those by Young [17], Kredell [7] and Niethammer [9], for $p = 2$, by Niethammer et al. [10], for $p = 3$, and by Hadjidimos, et al. [4], for any $p \geqslant 2$, in both the nonnegative and nonpositive cases. It should be mentioned that all of the works on the domains of convergence were concerned with p-cyclic consistently ordered matrices only. Also that the results in [4] were recovered by Wild and Niethammer [16] and, independently, by Noutsos [11], who obtained parametric expressions for all the boundary curves involved.

The main motivation for the present work is to extend the study of the convergence domains of the SOR method in [4] to the case where A in (1.1) is p-cyclic but *not* consistently ordered.

For the reasons that are explained and become clear in [3], in this manuscript we study the case where A^{T} is p-cyclic consistently ordered or, equivalently, when in Eq. (1.5), $q = 1$. In such a case the block Jacobi matrix associated with A has the following block form:

$$B := \begin{bmatrix} 0 & B_1 & 0 & \dots & 0 & 0 \\ 0 & 0 & B_2 & \dots & 0 & 0 \\ \vdots & & & & & \\ 0 & 0 & 0 & \dots & 0 & B_{p-1} \\ B_p & 0 & 0 & \dots & 0 & 0 \end{bmatrix}, \tag{1.6}$$

with its diagonal blocks being square. In this work we completely determine the regions of convergence by a recursive algorithm in both the nonpositive and the nonnegative cases. To accomplish it we use the Schur-Cohn algorithm [5] as this was done in [4] but now the analysis is much more complicated, due to the nature of the problem, and also more complete. In Section 2 the exact SOR convergence domains are derived in the general case $p \geqslant 3$ and the corresponding domains for the cases $p = 3, 4$ and 5 are completely studied and determined. An astonishing result obtained is that in the nonpositive case and for p odd there exist pairs $(\rho(B), \omega)$ for which the SOR method, in the nonconsistently ordered case of A we examine, converges while the SOR, in the consistently ordered case, corresponding to A^{T} does *not* (see Theorem 2.10). In Section 3 the Schur–Cohn algorithm is applied again to determine when the optimal value of the parameter ω is equal to 1 or, if it is not, to determine an interval in which the optimal ω lies. To the best of our knowledge, this method of obtaining information about the optimal ω by means of the Schur–Cohn algorithm is done for the first time in the literature.

2. Domains of convergence

2.1. *The Schur–Cohn algorithm*

One of the main tools in our analysis is the Schur–Cohn algorithm (see [5]) which is presented below. For this let

$$P(z) := a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0, \quad n \geqslant 0, \tag{2.1}$$

be a polynomial of degree n with $a_j \in \mathbb{C}$, $j = 0(1)n$, and $a_j \neq 0$ for at least one j. The *reciprocal* polynomial $P^*(z)$ is defined by

$$P^*(z) := \bar{a}_0 z^n + \bar{a}_1 z^{n-1} + \cdots + \bar{a}_n, \tag{2.2}$$

where $\bar{a}_j$ is the complex conjugate of a_j, $j = 0(1)n$, and satisfies

$$P^*(z) := z^n\overline{P(1/\bar{z})}. \tag{2.3}$$

We introduce the polynomial $TP(z)$ (or simply TP) of degree $n-1$ defined by

$$TP(z) := \bar{a}_0 P(z) - a_n P^*(z) = \sum_{k=0}^{n-1} (\bar{a}_0 a_k - a_n \bar{a}_{n-k}) z^k \tag{2.4}$$

which is called the Schur transform of $P(z)$. The iterated Schur transforms $T^2P, T^3P, \ldots, T^nP$ are defined by induction. We set now

$$\gamma_k := T^k P(0), \quad k = 1(1)n, \tag{2.5}$$

and give the Schur Theorem (see [5]).

Theorem 2.1. *Let $P := P(z)$ be a polynomial of degree n with $P \not\equiv 0$. All zeros of P lie outside the closed unit disk, $\bar{D}_1$, if and only if*

$$\gamma_k > 0, \quad k = 1(1)n. \tag{2.6}$$

2.2. *The nonpositive case*

Let all the eigenvalues μ of the block Jacobi matrix B in (1.6) satisfy

$$\mu^p \leqslant 0, \quad \mu \in \sigma(B). \tag{2.7}$$

Then the eigenvalues λ of the associated SOR matrix $\mathcal{L}_\omega$ will satisfy (1.5) with $q = 1$, namely

$$(\lambda + \omega - 1)^p = \mu^p \omega^p \lambda. \tag{2.8}$$

As was considered in [3] and for the reasons explained there, let v be any fixed but otherwise arbitrary number in the interval $[0, \rho(B)]$. For each such $v \in [0, \rho(B)]$ we will determine the interval, in terms of ω, in the (v, ω)-plane for which all the roots λ_j, $j = 1(1)p$, of (2.8) belong to the open unit disk, D_1, in the complex plane. Then we will determine the domain of convergence of the SOR method by considering the set of all possible values of $v \in [0, \rho(B)]$.

For $v = 0$, (2.8) gives $\lambda = 1 - \omega$ implying $|\lambda| < 1$ for all $\omega \in (0, 2)$. It is then obvious, using continuity arguments, that for $v = \varepsilon \to 0^+$, there will be an interval for ω, subinterval of $(0, 2)$, for which λ_j, $j = 1(1)n$, of (2.8) will satisfy $|\lambda_j| < 1$. For a certain $v \in (0, \rho(B)]$, (2.8) will become

$$(\lambda + \omega - 1)^p = -v^p \omega^p \lambda. \tag{2.9}$$

We set $\lambda = |\lambda| \mathrm{e}^{\mathrm{i}\phi}$, extract pth roots of both members of (2.9) to obtain

$$|\lambda| \mathrm{e}^{\mathrm{i}\phi} + \omega - 1 = v\omega |\lambda|^{1/p} \mathrm{e}^{(\mathrm{i}(2k+1)\pi + \phi)/p}, \quad k = 0(1)p - 1, \tag{2.10}$$

and put

$$z := |\lambda|^{1/p} \mathrm{e}^{(\mathrm{i}(2k+1)\pi + \phi)/p} \tag{2.11}$$

to produce

$$z^p + v\omega z + 1 - \omega = 0. \tag{2.12}$$

Hence, all the roots of the polynomial equation (2.12) must lie in the open unit disk D_1 or, equivalently, the zeros of the corresponding reciprocal polynomial must lie outside $\overline{D}_1$. From the discussion so far it is evident that one can apply the Schur Theorem with

$$P(z) := (1-\omega)z^p + v\omega z^{p-1} + 1 \tag{2.13}$$

and

$$P^*(z) := z^p + v\omega z + 1 - \omega. \tag{2.14}$$

Using (2.4) one readily obtains that

$$TP(z) := v\omega z^{p-1} - (1-\omega)v\omega z + \omega(2-\omega). \tag{2.15}$$

At this point we observe that $\omega \in (0,2)$ is a common factor in all three terms in (2.15). So, without loss of generality, and in order to simplify the analysis, instead of (2.15) we consider

$$TP(z) := vz^{p-1} - (1-\omega)vz + 2 - \omega. \tag{2.16}$$

(Note: In fact, (2.16) could have been obtained if instead of (2.13) we had considered $(1/\sqrt{\omega})P(z)$.)

By successive applications of the Schur transform to (2.16) we finally obtain

$$\begin{aligned} T^jP(z) &:= B_2^{(j)}z^{p-j} + B_1^{(j)}z + B_0^{(j)}, \quad j = 1(1)p-1, \\ T^pP(z) &:= [B_0^{(p-1)}]^2 - [B_1^{(p-1)} + B_2^{(p-1)}]^2. \end{aligned} \tag{2.17}$$

The coefficient sequences in (2.17) are derived from the recurrence relationships

$$B_2^{(j+1)} = -B_2^{(j)}B_1^{(j)}, \quad B_1^{(j+1)} = B_1^{(j)}B_0^{(j)}, \quad B_0^{(j+1)} = [B_0^{(j)}]^2 - [B_2^{(j)}]^2, \quad j = 1(1)p-2, \tag{2.18}$$

with initial values

$$B_2^{(1)} = v, \quad B_1^{(1)} = (\omega - 1)v, \quad B_0^{(1)} = 2 - \omega. \tag{2.19}$$

The values γ_j of the Schur Theorem, which must be positive, are then given by

$$\gamma_j := T^jP(0) = \begin{cases} B_0^{(j)}, & j = 1(1)p-1, \\ [B_0^{(p-1)}]^2 - [B_1^{(p-1)} + B_2^{(p-1)}]^2, & j = p. \end{cases} \tag{2.20}$$

Therefore, the SOR convergence domain we are seeking will be given by

$$\Omega_p := \{(\rho(B),\omega) | \gamma_j > 0, \ j = 1(1)p, \ \forall v \in [0, \rho(B)]\}. \tag{2.21}$$

As in [4], we introduce the quantities

$$\tilde{\gamma}_j := [B_0^{(j-1)}]^2 - [B_1^{(j-1)} + B_2^{(j-1)}]^2, \quad j = 2(1)p, \tag{2.22}$$

which will be very useful in the sequel.

Since for $\omega = 1$ direct conclusions can be drawn from (2.12) in what follows we may distinguish the cases $0 < \omega < 1$ and $1 < \omega < 2$. Below, a number of statements in the form of lemmas and theorems are given and proved which, eventually, lead us to the determination of the regions Ω_p in (2.21).

Lemma 2.2. *For $\omega < 1$, if $\gamma_j > 0$ for all $j = 1(1)p-1$, then $B_2^{(j)} > 0$ and $B_1^{(j)} < 0$ for all $j = 1(1)p-1$. On the other hand, for $\omega > 1$ if $\gamma_j > 0$ for all $j = 1(1)p-1$, then $B_2^{(j)} = (-1)^{j-1}|B_2^{(j)}|$ and $B_1^{(j)} > 0$ for all $j = 1(1)p-1$.*

Proof. For $\omega < 1$, from (2.19) we have that $B_2^{(1)} > 0$ and $B_1^{(1)} < 0$ while for $\omega > 1$ it is $B_2^{(1)} > 0$ and $B_1^{(1)} > 0$. In both cases our assertions can be very easily proved by induction using the relationships in (2.18). □

Lemma 2.3. *For all $j = 1(1)p-1$, $\gamma_j > 0$ if and only if (iff) $B_0^{(j-1)} + B_2^{(j-1)} > 0$ and $B_0^{(j-1)} - B_2^{(j-1)} > 0$.*

Proof. (2.20) and (2.18) imply that

$$\gamma_j = [B_0^{(j-1)} + B_2^{(j-1)}]\ [B_0^{(j-1)} - B_2^{(j-1)}]. \tag{2.23}$$

Let $\gamma_j > 0$, $j = 1(1)p-1$. Since $B_0^{(j-1)} = \gamma_{j-1} > 0$, we have from Lemma 2.2 that one of the two factors in (2.23) must be positive and so must be the other one. The converse holds in view of (2.23). It is noted that the proof just given does not cover the case $j = 1$. However, if one uses (2.13) and considers that $B_2^{(0)} = 1-\omega$, $B_1^{(0)} = \nu\omega$, $B_0^{(0)} = 1$, then obviously $B_0^{(0)} + B_2^{(0)} = 2-\omega > 0$ and $B_0^{(0)} - B_2^{(0)} = \omega > 0$. Therefore, our statement holds for all j's. □

Lemma 2.4. *For $\omega < 1$, if $\gamma_j > 0$ for all $j = 1(1)p-1$, then $\tilde{\gamma}_j > 0$ for all $j = 2(1)p$.*

Proof. From (2.22) and (2.18) and for any $j = 3(1)p$ it can be obtained that

$$\begin{aligned}\tilde{\gamma}_{j-1}\tilde{\gamma}_j = {} & [B_0^{(j-2)} - B_2^{(j-2)}]^2[B_0^{(j-2)} + B_2^{(j-2)} + B_1^{(j-2)}]^2 \\ & \times[B_0^{(j-2)} + B_2^{(j-2)} - B_1^{(j-2)}]\ [B_0^{(j-2)} - B_2^{(j-2)} - B_1^{(j-2)}].\end{aligned} \tag{2.24}$$

By virtue of Lemmas 2.2 and 2.3 neither of the first two factors on the right-hand side of (2.24) can be zero while both last factors are positive. So is then the product $\tilde{\gamma}_{j-1}\ \tilde{\gamma}_j$. By induction, it is readily seen that if $\tilde{\gamma}_j > 0$ for precisely one j, then $\tilde{\gamma}_j > 0$ for all $j = 2(1)p$. However, $\tilde{\gamma}_2 > \gamma_2 (> 0)$ as is easily checked which completes the proof. □

Lemma 2.5. *For $\omega > 1$, if $\gamma_j > 0$ for all $j = 1(1)p-1$, then for j odd it is $\tilde{\gamma}_j > 0$ while for j even it is $\tilde{\gamma}_j > 0$ iff $\tilde{\gamma}_2 > 0$.*

Proof. From (2.22) and (2.18) it can be obtained that

$$\tilde{\gamma}_j = [B_0^{(j-2)} - B_2^{(j-2)}]^2\ [B_0^{(j-2)} + B_2^{(j-2)} - B_1^{(j-2)}]\ [B_0^{(j-2)} + B_2^{(j-2)} + B_1^{(j-2)}]. \tag{2.25}$$

In view of Lemmas 2.2 and 2.3 it is obvious that $\mathrm{sign}(\tilde{\gamma}_j) = \mathrm{sign}(B_0^{(j-2)} + B_2^{(j-2)} - B_1^{(j-2)}) = \mathrm{sign}(B_0^{(j-3)} - B_2^{(j-3)} - B_1^{(j-3)})$. Similarly, $\mathrm{sign}(\tilde{\gamma}_{j-2}) = \mathrm{sign}(B_0^{(j-3)} - B_2^{(j-3)} - B_1^{(j-3)})$. Therefore, by induction, it is concluded that for j odd, $\tilde{\gamma}_j > 0$ iff $\tilde{\gamma}_3 > 0$ while for j even, $\tilde{\gamma}_j > 0$ iff $\tilde{\gamma}_2 > 0$. Using (2.19) and (2.22) it can be checked that $\tilde{\gamma}_3 > 0$ which completes the proof. □

Based on the results obtained so far one can give the following equivalent definitions for the convergence domains Ω_p in (2.21). This is done in the theorem below.

Theorem 2.6. *The convergence domain Ω_p in* (2.21) *can be equivalently given by*

$$\Omega_p = \{(\rho(B), \omega) | \gamma_j > 0, \quad j = 1(1)p-1, \ \forall v \in [0, \rho(B)]\}, \qquad p \text{ odd}, \tag{2.26}$$

and

$$\begin{aligned} \Omega_p = \{(\rho(B), \omega) | \ \gamma_j > 0, \quad j = 1(1)p-1, \quad \text{and} \\ \omega \in (0,1] \ \cup \left\{ \left(1, \frac{2}{1+v}\right) \text{ iff } v < 1 \right\}, \ \forall\, v \in [0, \rho(B)]\}, \qquad p \text{ even}. \end{aligned} \tag{2.27}$$

Proof. The conditions that define Ω_p in (2.21) are equivalent to

$$\gamma_j > 0, \quad j = 1(1)p-1, \quad \text{and} \quad \tilde{\gamma}_p > 0. \tag{2.28}$$

For odd p, $\tilde{\gamma}_p > 0$ in view of Lemmas 2.4 and 2.5. This proves (2.26). By virtue of the two previous lemmas for even p, $\tilde{\gamma}_p > 0$ in (2.28) is equivalent to $\tilde{\gamma}_2 > 0$. However, the latter inequality is always true for $\omega \leqslant 1$ while for $\omega > 1$ it is equivalent to $\omega < 2/(1+v)$, which holds iff $v < 1$. This proves (2.27). □

The following two statements enable us to determine orderings of the domains Ω_p.

Lemma 2.7. *The "right" boundaries $\partial\Omega_p$ of the domains Ω_p defined in* (2.26) *and* (2.27) *are given by the "leftmost" of the curves c_p, where*

$$c_p := \{(v, \omega) | \gamma_{p-1} = 0, \ \omega \in (0,2)\}, \qquad p \text{ odd}, \tag{2.29}$$

and

$$c_p := \left\{ (v, \omega) | \gamma_{p-1} = 0, \ \omega \in (0,1] \text{ for } v \geqslant 1 \text{ and } \omega = \frac{2}{1+v} \text{ for } v \leqslant 1 \right\}, \qquad p \text{ even}. \tag{2.30}$$

Proof. From (2.26) and (2.27) it is seen that the curves $\gamma_j = 0$, $j = 1(1)p-1$, are right boundary curves for the domain Ω_p. However, (2.18) gives

$$\gamma_{j+1} = \gamma_j^2 - [B_2^{(j)}]^2. \tag{2.31}$$

Let $(\bar{v}, \bar{\omega})$ be any point such that $\gamma_j(\bar{v}, \bar{\omega}) = 0$. From (2.31), $\gamma_{j+1}(\bar{v}, \bar{\omega}) \leqslant 0$. Therefore, the curve $\gamma_{j+1} = 0$ is a "better" bound than $\gamma_j = 0$. Use of induction completes the proof. □

Theorem 2.8. *For the ordering of the domains Ω_p of Theorem* 2.6 *there hold*

$$\Omega_{p+2} \subset \Omega_p, \qquad p = 2, 3, 4, \ldots, \tag{2.32}$$

and

$$\Omega_{p+1} \subset \Omega_p, \qquad p = 3, 5, 7, \ldots. \tag{2.33}$$

Proof. From Theorem 2.6 it is obvious that

$$\Omega_{p+2} \subseteq \Omega_p, \quad p = 2,3,4,\ldots. \tag{2.34}$$

To prove the validity of (2.32) it suffices to show that the curves $\gamma_j = 0$ and $\gamma_{j+1} = 0$ for $j = 1(1)p-2$ are not identically the same or, equivalently, in view of (2.31), $B_2^{(j)} \not\equiv 0$. However, by induction, using (2.18) we have

$$B_2^{(j)} = B_0^{(j-2)}[B_0^{(j-3)}]^2 \ldots [B_0^{(1)}]^{j-2}[-B_1^{(1)}]^{j-2}B_2^{(1)}. \tag{2.35}$$

Since $B_0^{(k)} = \gamma_k > 0$, $k = 1(1)j-2$, $B_1^{(1)} = (\omega - 1)v$ and $B_2^{(1)} = v$ we have from (2.35) that $B_2^{(j)} = 0$ holds iff $\omega = 1$ or $v = 0$. Consequently, $B_2^{(j)} \not\equiv 0$ and (2.32) is proved. From the definitions (2.26) and (2.27) of Theorem 2.6 we readily obtain that

$$\Omega_{p+1} \subseteq \Omega_p, \quad p = 3,5,7,\ldots. \tag{2.36}$$

That strict inclusion holds in (2.36) follows by the same reasoning as in the proof of (2.32) since the two curves $\gamma_{p-1} = 0$ and $\gamma_p = 0$ for $\omega \in (0,1]$ do not coincide. □

From the analysis so far it becomes clear that the right boundary curve $\partial\Omega_p$ can always be expressed as a single-valued function of ω, $v = v_p(\omega)$, $\omega \in (0,2)$. $\partial\Omega_p$ can also be expressed as a single-valued function of v, $\omega = \omega_p(v)$, $v \in [0, \rho(B)]$, if it is *strictly* decreasing. As will be seen in the sequel this is the case for $p = 3,4$ and 5, where explicit expressions for $\partial\Omega_p$ are derived. For $p > 5$ this issue requires further investigation.

(i) $p = 3$: From $\gamma_2 = 0$ and relationships (2.18) and (2.19) we have that

$$\gamma_2 = (2 - \omega - v)(2 - \omega + v) = 0$$

or, equivalently, $2 - \omega - v = 0$ implying $\omega = 2 - v$. Since this equality must hold for all $v \in [0, \rho(B)]$ it is readily concluded that

$$\omega_3 := \omega_3(v) = 2 - v, \qquad v = \rho(B) < 2. \tag{2.37}$$

The convergence domain Ω_3 is illustrated in Fig. 1.

Note: It is interesting to note that for $v \in (0,1)$ there are more pairs (v,ω) for which the SOR considered in the present paper converges than in the corresponding case where A is p-cyclic consistently ordered. These are all the pairs (v,ω), $v \in (0,1)$, between the dotted line (included) and the solid line (excluded) in Fig. 1. This conclusion constitutes a very special case of a more general one (see Theorem 2.10).

(ii) $p = 4$: From $\gamma_3 = 0$ and relationships (2.18) and (2.19) we obtain

$$[(2 - \omega - v)(2 - \omega + v) + (1 - \omega)v^2]\ [(2 - \omega - v)(2 - \omega + v) - (1 - \omega)v^2] = 0.$$

The fact that $\gamma_2 > 0$ together with $0 < \omega \leqslant 1$ imply that the first factor in the previous product is positive. Therefore, $(2 - \omega - v)(2 - \omega + v) - (1 - \omega)v^2 = 0$ or, equivalently, $\omega = 2 - v^2$. For this equation to hold for all $v \in [0, \rho(B)]$ we must have

$$\omega_4 := \omega_4(v) = 2 - v^2, \ v = \rho(B) < \sqrt{2},\ 0 < \omega \leqslant 1. \tag{2.38}$$

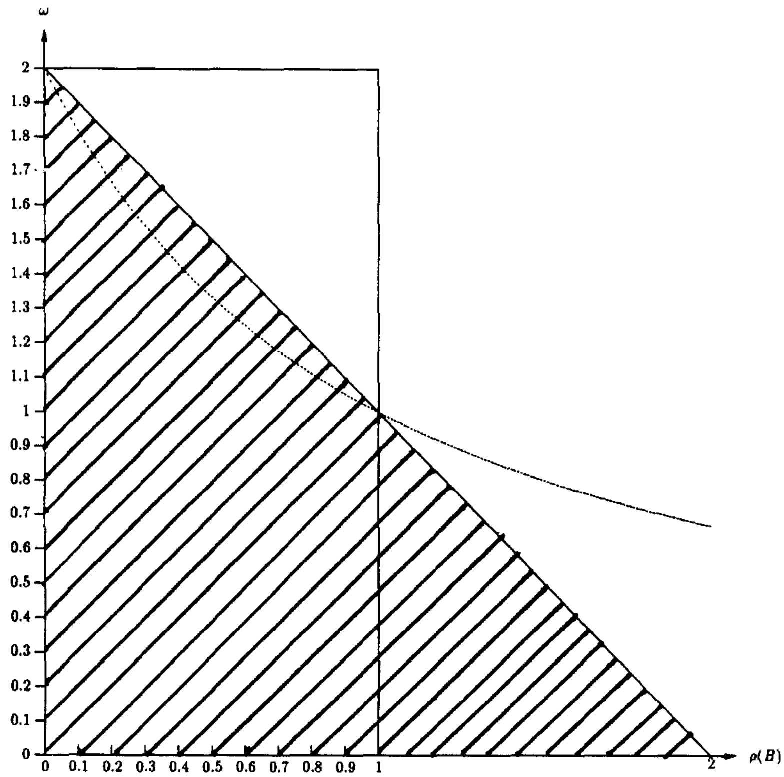

Fig. 1. Nonpositive case $p = 3$.

For $1 \leqslant \omega < 2$ we already have $\omega = 2/(1+v)$ and for all $0 \leqslant v \leqslant \rho(B)$,

$$\omega_4 := \omega_4(v) = \frac{2}{1+v}, \quad v = \rho(B) < 1 \leqslant \omega < 2. \tag{2.39}$$

The union of the two curves (2.38) and (2.39) gives the right boundary of $\partial\Omega_4$. The domain Ω_4 is illustrated in Fig. 2.

(iii) $p = 5$: From $\gamma_4 = 0$, (2.18), (2.19), after some simple algebra we have that

$$f(v,\omega) := \omega^2 + (v^3 + v - 4)\omega + 4 - 2v - v^2 = 0. \tag{2.40}$$

If for $v \in [0, \rho(B)]$ there exists a right boundary curve $\omega = \omega_5(v)$ of Ω_5, the ω in question will be obtained as a solution from (2.40). However, from Theorem 2.6, or from Lemma 2.7, it is implied that $\omega_5(v)$ must be to the "left" ("below") the curve $\omega = 2 - v$ of the case $p = 3$ and "above" the v-axis ($\omega = 0$). It can be readily checked that $f(2-v, v) = -v^2(v-1)^2 < 0$, for $v \in (0, \rho(B)] \setminus \{1\}$, while $f(+\infty, v) = +\infty > 0$. Hence, one of the two *real* roots of (2.40) is not admissible. For the other one to be admissible $f(0,v) = 4 - 2v + v^2 > 0$ must hold. This inequality holds for all $v \in [0, -1+\sqrt{5}]$. Consequently, $\omega_5(v)$ exists and is given by the smaller zero of (2.40), namely

$$\omega_5(v) = \frac{4 - v - v^3 - \sqrt{(4 - v - v^3)^2 - 4(4 - 2v - v^2)}}{2}, \quad 0 \leqslant v < -1 + \sqrt{5}. \tag{2.41}$$

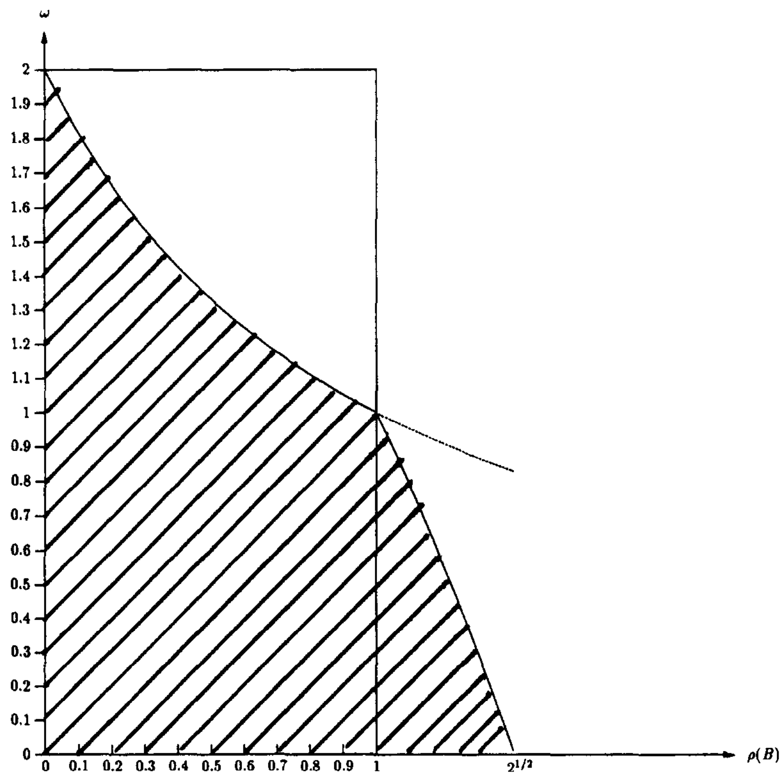

Fig. 2. Nonpositive case $p = 4$.

For $v \in [0, \rho(B)] \subset [0, -1+\sqrt{5})$ we differentiate (2.41) to obtain

$$\operatorname{sign}\left(\frac{\mathrm{d}\omega}{\mathrm{d}v}\right) = \operatorname{sign}\Big(-[(v^3+v-4)(3v^2+1)+4(v+1)] - (3v^2+1)\sqrt{(v^3+v-4)^2+4(v^2+2v-4)}\Big). \tag{2.42}$$

Now, if $K \equiv (v^3+v-4)(3v^2+1)+4(v+1) = v(v-1)(3v^3+3v^2+7v-5) > 0$ which is true for $v \in [0, v_0) \cup (1, -1+\sqrt{5})$ where $v_0 \approx 0.530$, the unique positive root of $3v^3+3v^2+7v-5=0$, then from (2.41), $\mathrm{d}\omega/\mathrm{d}v < 0$. If, on the other hand, $v \in [v_0, 1]$ then $K \leqslant 0$ and (2.42) can be equivalently written as

$$\begin{aligned}\operatorname{sign}\left(\frac{\mathrm{d}\omega}{\mathrm{d}v}\right) &= (K^2-(3v^2+1)^2[(v^3+v-4)^2+4(v^2+2v-4)])\\ &= \operatorname{sign}\left(v^2(v-1)^2(-3v^2-18v+5)\right).\end{aligned}$$

However, as is readily checked, $-3v^2-18v+5 \leqslant 0$ for $v \in [\frac{1}{3}(-9+4\sqrt{6}), \infty) \supseteq [v_0, 1]$, with equality holding for $v=1$. Thus again $\mathrm{d}\omega/\mathrm{d}v \leqslant 0$ and the function $\omega_5(v)$ in (2.41) is a strictly decreasing function of v. Therefore, the right boundary of Ω_5 is given by

$$\omega_5 := \omega_5(v) = \frac{4-v-v^3-\sqrt{(v^3+v-4)^2+4(v^2+2v-4)}}{2}, \quad v = \rho(B) < -1+\sqrt{5}. \tag{2.43}$$

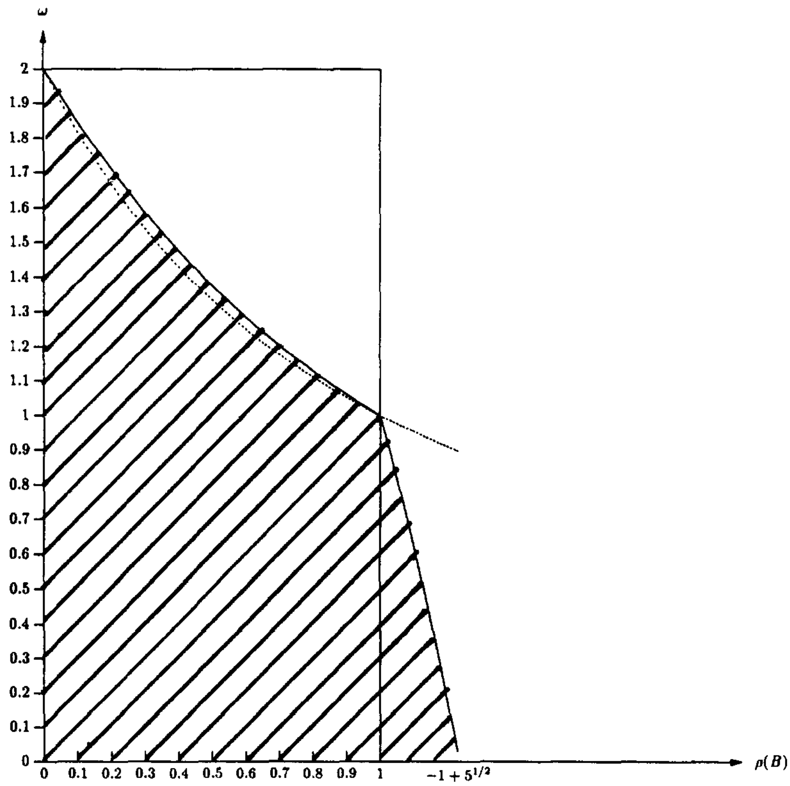

Fig. 3. Nonpositive case $p = 5$.

Fig. 3 depicts the domain Ω_5, where the same note as in the case $p = 3$ can be made.

In Theorem 2.8 orderings of the domains Ω_p were determined. A question that may arise is whether the sequence $\{\Omega_p\}_{p=3}^{\infty}$ converges to a limit and in case of convergence whether the limit in question can be found. That both subsequences $\{\Omega_p\}_{p=3,5,7,\dots}$ and $\{\Omega_p\}_{p=4,6,8,\dots}$ converge is obvious from (2.32). From (2.33), however, it is also obvious that for their limits there will hold $\lim_{p\to\infty} \Omega_{2p+2} \subseteq \lim_{p\to\infty} \Omega_{2p+1}$. In fact, the following statement (Theorem 2.9) similar to the corresponding one in [4] can be proved.

Theorem 2.9. *For the region*

$$\Omega := \left\{(v,\omega)|\ 0 \leqslant v \leqslant 1,\ 0 < \omega < \frac{2}{1+v}\right\}, \tag{2.44}$$

depicted in Fig. 4, *there holds*

$$\Omega \subseteq \bigcap\nolimits_{p=1}^{\infty} \Omega_{2p+2} \subseteq \bigcap\nolimits_{p=1}^{\infty} \Omega_{2p+1}. \tag{2.45}$$

Proof. The inclusion on the right of (2.45) follows from the previous discussion and the strictly decreasing character of the two subsequences. For the inclusion on the left it suffices to prove that all

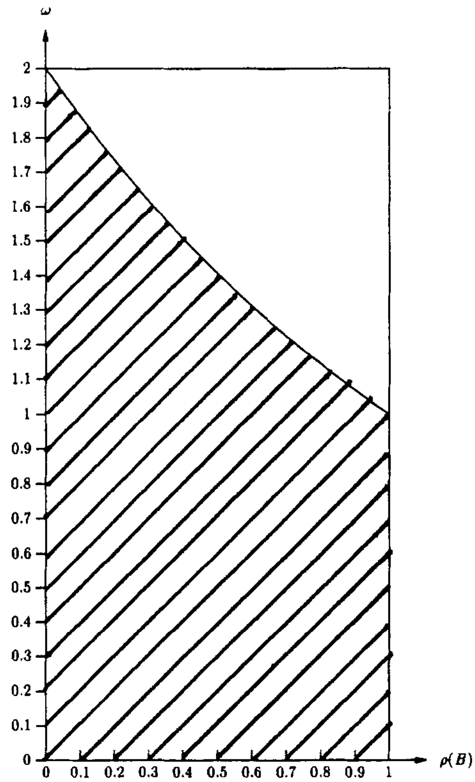

Fig. 4. Nonnegative case $p = 3, 5, 7, \ldots$.

the zeros of the polynomial (2.13) lie outside $\overline{D}_1$. Consider the polynomials $Q(z)$ and $R(z)$ defined by

$$Q(z) := (1-\omega)z^p, \quad R(z) := -v\omega z^{p-1} - 1, \tag{2.46}$$

so that their difference gives the polynomial $P(z)$ in (2.13). For $v = 0$ and $\omega \in (0,2)\setminus\{1\}$ the zeros of $P(z) = Q(z) - R(z)$ are given by $z = \sqrt[p]{1/(\omega-1)}$, hence $|z| > 1$, and the zeros of $P(z)$ lie outside $\overline{D}_1$. For $v = 0$ and $\omega = 1$, $P(z)$ has $z = 0$ as a pole of multiplicity p. For $v \neq 0$ the zeros of $R(z)$ lie on the circle with radius $\sqrt[p-1]{1/(v\omega)}$. Since $\omega < 2/(1+v) \leqslant 1/v$ the radius in question is greater than 1. Suppose now that $z \in \partial D_1$ (unit circle). Then it will be $z = x + \mathrm{i}y$, $x, y \in \mathbb{R}$, $x^2 + y^2 = 1$. For $v \neq 1$ we successively obtain

$$|Q(z)| = |1-\omega| < |1-\omega v| = |1-\omega v|z|^{p-1}| \leqslant |1+\omega v z^{p-1}| = |R(z)|, \tag{2.47}$$

where the strict inequality on the left holds because $\omega < 2/(1+v)$. In view of (2.47), the previous analysis and the fact that all the zeros of $R(z)$ lie outside $\overline{D}_1$, Rouché's Theorem (see [5] or [12]) implies that so do all the zeros of $P(z)$. The only case that has not been examined so far is that when $v = 1$. For $v = 1$, $\omega < 1$ and the inequality on the left of (2.47) becomes equality. Then, however, $P(z) = 0$ gives $z^{p-1}((1-\omega)z + \omega) = -1$ from which $1 = |(1-\omega)z + \omega|$. This implies that, $|(1-\omega)(x+iy)+\omega| = 1$ or, equivalently, $(1-\omega)(1-x) = 0$. Thus, $x = 1$ (and $y = 0$) so that $z = 1$. But the number $z = 1$ is not a zero of $P(z)$ as is readily checked, meaning that the particular case we have been examining cannot happen. This completes the proof. $\square$

Theorem 2.10. *Let $\hat{\Omega}_p$, $p = 3,5,7,\ldots$, be the region of convergence of the p-cyclic consistently ordered SOR method corresponding to A^{T} and let T be the open rectangle with vertices in the (v,ω)-plane $(0,0),(1,0),(1,2),(0,2)$. There exists a nonempty region Ψ_p, $p = 3,5,7,\ldots$, defined by*

$$\Psi_p = \Omega \cap T \backslash \hat{\Omega}_p, \quad p = 3,5,7,\ldots, \tag{2.48}$$

such that for any $(v,\omega) \in \Psi_p$ the SOR method for the matrix A studied so far converges while the corresponding SOR for the consistently ordered matrix A^{T} diverges.

Proof. Having in mind the upper right boundary of the SOR region of convergence in the consistently ordered case, which is given by $\omega = 2/(1+v)$ (see [4]), to prove our assertion it suffices to prove that the points $(v,\omega) = (v, 2/(1+v))$, with $0 < v < 1$, lie strictly within the SOR region of convergence of the present nonconsistently ordered case for every $p = 3,5,7,\ldots$. For this we consider as in Theorem 2.9 the polynomial $P(z) = Q(z) - R(z)$, with $Q(z)$ and $R(z)$ being defined in (2.46). The zeros of $R(z)$ lie again on the circle with radius $\sqrt[p-1]{1/(v\omega)}$. This time it is $\omega = 2/(1+v) < 1/v$ and the radius of the circle in question is again greater than 1. For $|z| = 1$ the notation is similar to but the analysis is different from that in the corresponding part of Theorem 2.9. This time in (2.47) the first strict inequality becomes equality since $\omega = 2/(1+v)$. If the second inequality in (2.47) were equality then equating the second leftmost and rightmost terms of equalities (2.47) and using the expression for ω, we would obtain, after some manipulation, that $\mathrm{Re}(z^{p-1}) = -1$. However, since $|z| = 1$, if $z = \cos\phi + \mathrm{i}\sin\phi$ were the polar form of z then from $z^{p-1} = \cos(p-1)\phi + \mathrm{i}\sin(p-1)\phi = -1$ we would have $\phi = (2q+1)\pi/(p-1)$, $q = 0,1,2,\ldots,p-2$. Also it would be $z^p = -z = -\cos((2q+1)\pi/(p-1)) - \mathrm{i}\sin((2q+1)\pi/(p-1))$. But then in the expression for the polynomial $P(z)$ there would be a complex number coming from its first term with imaginary part $\mathrm{Im}\,P(z) = (\omega - 1)\sin((2q+1)\pi/(p-1)) \neq 0$. This is because $2q+1$ is odd and $p-1$ even and as a result the argument involved in the previous expression cannot become an integral multiple of π making, in turn, possible for $\mathrm{Im}\,P(z)$ to become zero. Consequently, z, with $|z| = 1$, cannot be a zero of $P(z)$ which concludes the proof. $\square$

Remarks. (i) The domain Ω of Theorem 2.9 is nothing but the domain S associated with the p-cyclic consistently ordered case ($q = p-1$) considered in [4].

(ii) In [4] the corresponding sequence of $\{\Omega_p\}_{p=3}^{\infty}$ was strictly decreasing and had as a limit the domain Ω ($\equiv S$) of Theorem 2.9.

(iii) The result in (ii) previously was obtained in [4] because the optimal values for the relaxation factor ω had been available (in the p-cyclic consistently ordered case). This vital information we lack in the present case since the corresponding optimal values have been found only for $p = 3$ and 4 (see [3]). However, it is conjectured that the leftmost inclusion in (2.45) is a strict set equality; on the other hand, in view of Theorem 2.10, it is implied that the rightmost one is a strict set inclusion.

(iv) For $\omega = 0$ the maximum admissible values of v ($= \rho(B)$) in the general case we have been examining are the same as those in [4]. To see this let $D_2^{(j)}$, $D_1^{(j)}$, and $D_0^{(j)}$ be the values of $B_2^{(j)}(\omega = 0)$, $B_1^{(j)}(\omega = 0)$ and $B_0^{(j)}(\omega = 0)$, respectively. It is obtained that $D_2^{(1)} = \rho(B)$, $D_1^{(1)} = -\rho(B)$ and $D_0^{(1)} = 2$. The corresponding values in [4] are $C_2^{(1)} = -\rho(B)$, $C_1^{(1)} = -\rho(B)$, and $C_0^{(1)} = 2$, respectively. By induction it is easily shown that $D_0^{(j)} = C_0^{(j)}$, $j \geqslant 2$, proving our assertion. $\square$

2.3. The nonnegative case

Our starting point is again Eq. (2.9). Working in a way similar to the one in the previous case the following polynomial equation is produced:

$$z^p - \nu\omega z + \omega - 1 = 0. \tag{2.49}$$

The Schur–Cohn algorithm is applied again with

$$B_2^{(1)} = -\nu\omega, \qquad B_1^{(1)} = (\omega - 1)\nu\omega, \qquad B_0^{(1)} = \omega(2-\omega), \tag{2.50}$$

or, simplifying by ω as before, with

$$B_2^{(1)} = -\nu, \qquad B_1^{(1)} = (\omega - 1)\nu, \qquad B_0^{(1)} = 2-\omega. \tag{2.51}$$

As in the nonpositive case we give a number of statements some of which are presented without proof in case their proof is similar to the corresponding one of Section 2.2.

Lemma 2.11. *For $\omega < 1$, if $\gamma_j > 0$ for all $j = 1(1)p-1$, then $B_2^{(j)} < 0$ and $B_1^{(j)} < 0$ for all $j = 1(1)p-1$; while for $\omega > 1$, if $\gamma_j > 0$ for all $j = 1(1)p-1$ then $B_2^{(j)} = (-1)^j|B_2^{(j)}|$ and $B_1^{(j)} > 0$ for all $j = 1(1)p-1$.*

Proof. Analogous to that of Lemma 2.2. □

Lemma 2.12. *For all $j = 1(1)p-1$, $\gamma_j > 0$ iff $B_0^{(j-1)} + B_2^{(j-1)} > 0$* and *$B_0^{(j-1)} - B_2^{(j-1)} > 0$.*

Proof. Analogous to that of Lemma 2.3. □

Lemma 2.13. *For $\omega < 1$, if $\gamma_j > 0$ for all $j = 1(1)p-1$, then $\tilde{\gamma}_j > 0$ for all $j = 2(1)p$ iff $\nu < 1$.*

Proof. In a similar way to that in Lemma 2.4 it can be proved that $\tilde{\gamma}_j > 0$ iff $\tilde{\gamma}_2 > 0$. From (2.51) and (2.22) it can be readily found out that $\tilde{\gamma}_2 > 0$ iff $\nu < 1$. □

Lemma 2.14. *For $\omega > 1$, if $\gamma_j > 0$ for all $j = 1(1)p-1$, then for j even it is $\tilde{\gamma}_j > 0$ while for j odd it is $\tilde{\gamma}_j > 0$ iff $\omega < 2/(1+\nu)$.*

Proof. In an analogous way to that in Lemma 2.5 it is proved that $\tilde{\gamma}_j > 0$ iff $\tilde{\gamma}_{j-2} > 0$. By induction we have that for j even, $\tilde{\gamma}_j > 0$ iff $\tilde{\gamma}_2 > 0$ which is valid since $\tilde{\gamma}_2 > \gamma_2 > 0$. For j odd, $\tilde{\gamma}_j > 0$ iff $\tilde{\gamma}_3 > 0$ which is equivalent to $\omega < 2/(1+\nu)$. □

Theorem 2.15. *The convergence domain Ω_p in (2.21) can be equivalently given by*

$$\Omega_p := \left\{(\rho(B),\omega)|\ \gamma_j > 0,\ \ j = 1(1)p-1,\ \omega < \frac{2}{1+\nu},\ \ \forall \nu \leqslant \rho(B) < 1\right\},\quad p\ odd, \tag{2.52}$$

and

$$\Omega_p := \{(\rho(B),\omega)|\ \gamma_j > 0,\ \ j = 1(1)p-1,\ \ \forall \nu \leqslant \rho(B) < 1\},\quad p\ even. \tag{2.53}$$

Proof. The proof is obvious since it is an immediate consequence of Lemmas 2.13 and 2.14. □

Lemma 2.16. *The "right" boundaries $\partial\Omega_p$ of the domains Ω_p defined in* (2.52) *and* (2.53) *can be equivalently given by the "leftmost" curves c_p where*

$$c_p := \left\{(v,\omega)|\ \gamma_{p-1} = 0,\ v = 1 \ for\ \omega \in (0,1],\ \omega = \frac{2}{1+v}\ for\ v < 1\right\}, \quad p\ odd, \tag{2.54}$$

and

$$c_p := \{(v,\omega)|\ \gamma_{p-1} = 0,\ v = 1\ for\ \omega \in (0,1]\}, \quad p\ even. \tag{2.55}$$

Proof. The proof is based on the definitions (2.52) and (2.53) and on a reasoning which duplicates that in the proof of Lemma 2.7. □

Theorem 2.17. *Let Ω' be defined by*

$$\Omega' := \Omega \setminus \{(v,\omega) \mid v = 1\}, \tag{2.56}$$

where Ω is the domain defined in (2.44) *of Theorem* 2.9. *Ω' is an SOR convergence domain for any $p \geqslant 3$.*

Proof. The proof is similar to that of Theorem 2.9, with the only difference being that the polynomials Q and R, defined in (2.46), are now defined by

$$Q(z) := (\omega - 1)z^p, \quad R(z) := v\omega z^{p-1} - 1. \quad □ \tag{2.57}$$

Theorem 2.18. *For the domains Ω_p of Theorem* 2.15 *there holds*

$$\Omega_p \equiv \Omega', \qquad p = 3,5,7,\ldots \tag{2.58}$$

and

$$\Omega_{p+2} \subset \Omega_p, \qquad p = 4,6,8,\ldots. \tag{2.59}$$

Proof. From (2.52) we have that the curves $v = 1$, for $\omega < 1$, and $\omega = 2/(1+v)$, for $\omega > 1$, are among those defining the boundaries of Ω_p, $p = 3,5,7,\ldots$, while at the same time they constitute boundary curves for Ω'. Hence, $\Omega_p \subseteq \Omega'$. However, since Ω' is a convergence domain of the SOR method for any $p \geqslant 3$ (see Theorem 2.17) it is implied that $\Omega' \subseteq \Omega_p$. These two inclusions imply (2.58). The proof of (2.59) is based on the definitions (2.53), on Theorem 2.17 and on a reasoning similar to that in the proof of Theorem 2.8. □

Theorem 2.19. *For the domain Ω' in* (2.56) *and the domains Ω_p in* (2.53) *there holds*

$$\Omega' \subseteq \bigcap_{p=1}^{\infty} \Omega_{2p+2}. \tag{2.60}$$

We simply note that the domains Ω_p, $p = 3,5,7,\ldots$, coincide almost with Ω', that is, they are those shown in Fig. 4 except for the line segment $v = 1$, $0 < \omega \leqslant 1$ which is not included.

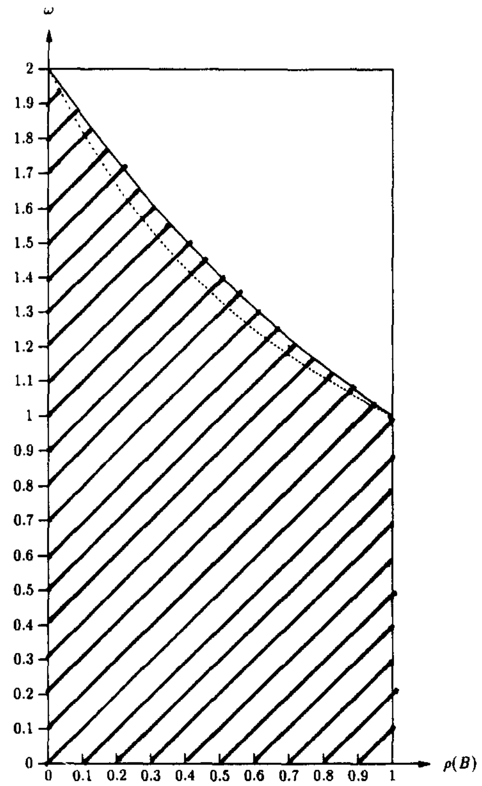

Fig. 5. Nonnegative case $p = 4$.

To determine each $\Omega_p, p = 4, 6, 8, \ldots$, one has to work in a recursive manner. So, the question arising just after Theorem 2.8 remains an open one in the general case $p \geqslant 6$, p even. However, by following a reasoning similar to that in Theorem 2.10, it can be proved that the upper right part of the boundary is strictly above the curve $\omega = 2/(1+v)$, $0 < v < 1$.

In what follows we find the right boundary curve in the case $p = 4$.

$p = 4$: From Lemma 2.16 we have that this boundary is given by the "leftmost" parts of the curves $v = 1$ and $\gamma_3 = 0$. From relationships (2.18), (2.20) and (2.51) we have that

$$\gamma_3 = [(2-\omega)^2 - v^2]^2 - [v^2(\omega - 1)]^2.$$

Having in mind that $\gamma_2 > 0$ and $v \leqslant 1$ we readily obtain that

$$\omega_4 := \omega_4(v) := \tfrac{1}{2}(v^2 + 4 - v\sqrt{v^2+8}), \quad 0 \leqslant v \leqslant 1. \tag{2.61}$$

It can be found out that $d\omega_4/dv < 0$ implying that (2.61), with $v = \rho(B) < 1$, will give the "upper" right boundary of Ω_4. The region Ω_4 is illustrated in Fig. 5. The dotted line shows the curve $\omega = 2/(1+v)$.

3. On the optimal values of ω

3.1. Introduction

The Schur–Cohn algorithm used extensively in Section 2 to derive the convergence domains of the SOR method can also be used to decide whether the optimal value of the relaxation factor ω, denoted by $\hat{\omega}$, is such that $\hat{\omega} = 1$ or $\hat{\omega} \in (0, 1)$, or $\hat{\omega} \in (1, 2)$. In the subsequent analysis we examine again the nonpositive and the nonnegative cases. As will be seen, some new interesting results are obtained and some well-known ones are recovered.

3.2. The nonpositive case

We begin our analysis with (2.12) where $\omega = 1$, namely,

$$z^p + vz = 0. \tag{3.1}$$

Obviously, (3.1) has one root equal to zero while all its other $p-1$ roots are complex and lie on the circle with radius $v^{1/(p-1)}$. If there exists at least one value of $\omega \neq 1$ such that all the p roots of (2.12) have modulus strictly less that $v^{1/(p-1)}$ then the corresponding SOR iteration matrix will have spectral radius strictly less than $v^{1/(p-1)}$ and the optimal value for $\omega(\hat{\omega})$ will be different from 1. So, in what follows, we seek the conditions on ω under which all the moduli of the roots of (2.12) become smaller (or greater) than $v^{1/(p-1)}$.

For our study we make the transformation

$$z := v^{1/(p-1)}\zeta \tag{3.2}$$

so that (3.1) becomes

$$\zeta^p + \zeta = 0 \tag{3.3}$$

and the images of the roots of (3.1), that laid on the circle with radius $v^{1/(p-1)}$, lie now on the unit circle ∂D_1. However, under (3.2), (2.14) becomes

$$P^*(\zeta) := v^{p/(p-1)}\zeta^p + \omega v^{p/(p-1)}\zeta + 1 - \omega = 0 \tag{3.4}$$

and the associated reciprocal polynomial is

$$P(\zeta) := (1-\omega)\zeta^p + \omega v^{p/(p-1)}\zeta^{p-1} + v^{p/(p-1)}. \tag{3.5}$$

To examine under what conditions all the zeros of (3.5) lie strictly outside $\overline{D}_1$ the Schur–Cohn algorithm will be used. This time the associated values of $B_2^{(1)}$, $B_1^{(1)}$ and $B_0^{(1)}$ are given by

$$B_2^{(1)} = \omega v^{2p/(p-1)}, \quad B_1^{(1)} = (\omega - 1)\omega v^{p/(p-1)}, \quad B_0^{(1)} = v^{2p/(p-1)} - (1-\omega)^2 \tag{3.6}$$

while the values $B_2^{(j)}$, $B_1^{(j)}$ and $B_0^{(j)}$, $j = 2(1)p-1$ are given again by (2.18). Since the signs of the values $B_2^{(1)}$ and $B_1^{(1)}$ of (3.6) are the same as those of the corresponding values of (2.19) and since $B_0^{(1)} = \gamma_1$ is required to be positive, the theory developed in Section 2.2 holds in general.

First the case $\omega > 1$ is examined when the following theorem can be stated and proved.

Theorem 3.1. *For $v = \rho(B)$ the minimization of $\rho(\mathcal{L}_\omega)$ for all $\omega \geqslant 1$ is achieved for $\omega = 1$.*

Proof. For $p \geqslant 3$ and from (2.18) and (3.6) we have

$$\gamma_2 = [v^{2p/(p-1)} - (1-\omega)^2]^2 - \omega^2 v^{4p/(p-1)}. \tag{3.7}$$

It is readily seen that

$$\operatorname{sign}(\tilde{\gamma}_2) = \operatorname{sign}((1-\omega)(v^{2p/(p-1)} + \omega v^{p/(p-1)} + \omega - 1)) = \begin{cases} -1, & \omega > 1, \\ 0, & \omega = 1, \end{cases}$$

and the proof is complete. □

For $\omega \leqslant 1$ the two statements given in the sequel can be stated and proved.

Lemma 3.2. *The domain $\tilde{\Omega}_p$, defined in a way quite analogous to* (2.26)–(2.27), *for which all the zeros of the polynomial* (3.5) *lie outside $\overline{D}_1$, can be given by*

$$\tilde{\Omega}_p := \{(\rho(B), \omega) |\ \gamma_{p-1} > 0, \quad \forall v \in [0, \rho(B)]\}. \tag{3.8}$$

Proof. From the definition of $\tilde{\Omega}_p$ we have

$$\tilde{\Omega}_p := \{(\rho(B), \omega) | \gamma_j > 0,\ j = 1(1)p-1,\ \tilde{\gamma}_p > 0,\ \forall v \in [0, \rho(B)]\}. \tag{3.9}$$

From the proof of Lemma 2.7 it is implied that $\gamma_{p-1} > 0$ gives a subdomain of the domains given by $\gamma_j > 0$, $j = 1(1)p-2$. So, the intersection of all these domains is the subdomain defined by $\gamma_{p-1} > 0$. On the other hand, from the proof of Lemma 2.4, we have that $\tilde{\gamma}_p > 0$ iff $\tilde{\gamma}_2 > 0$. However, it can be checked that

$$\operatorname{sign}(\tilde{\gamma}_2) = \operatorname{sign}(B_0^{(1)} + B_2^{(1)} + B_1^{(1)}) = \operatorname{sign}(\gamma_1),$$

which completes the proof. □

Theorem 3.3. *For $\omega \leqslant 1$, if $(v, \omega) \in \tilde{\Omega}_p \cap \Omega_p$ for $v \in [0, \rho(B)]$ then the smallest $\rho(\mathcal{L}_\omega)$ corresponds to an optimal ω, $\hat{\omega} < 1$, otherwise it corresponds to $\hat{\omega} = 1$.*

Proof. If $(v, \omega) \in \tilde{\Omega}_p$ then our assertion holds by virtue of Lemma 3.2. Moreover, for convergence to be achieved there must be $(v, \omega) \in \Omega_p$ and the proof is complete. □

Remarks. (i) For $p = 2$, our theory is trivially verified and the well-known result obtained by Kredell [7] and Niethammer [2], namely $\hat{\omega} < 1$, is confirmed by our analysis. Of course, in [7, 2], an analytic expression for $\hat{\omega}$, namely $\hat{\omega} = 2/(1 + (1 + \rho^2(B))^{1/2})$, was also obtained.

(ii) For $p = 3, 4$ analogous results for $\hat{\omega}$ obtained in [3] are confirmed by the present analysis. Again we comment that analytic expressions for $\hat{\omega}$ were given in [3].

(iii) For the special case $p = 5$, equation $\gamma_4 = 0$ gives

$$\{[v^{5/2} - (1-\omega)^2]^2 - \omega^2 v^5\}^2 - (1-\omega)^2\omega^4 v^{15/2} - (1-\omega)^2\omega^3 v^5[v^{5/2} - (1-\omega)^2] = 0. \tag{3.10}$$

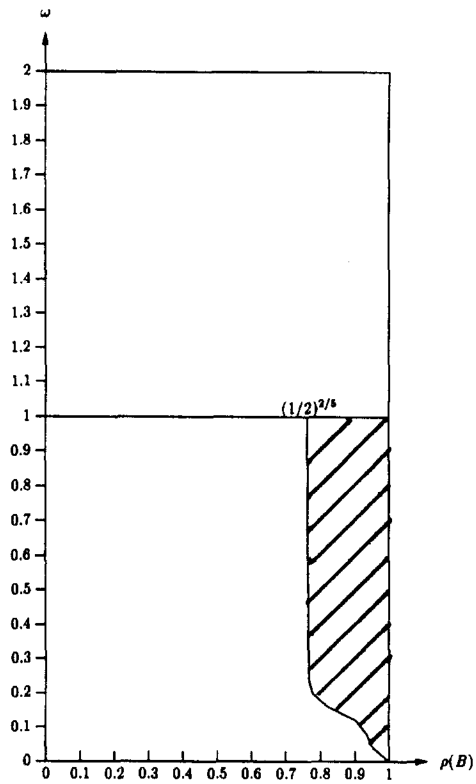

Fig. 6. Nonpositive case $p = 5$. Domain of optimal ω for $\rho(B) > (1/2)^{2/5}$.

As is readily checked, curve (3.10) passes through the points $(\nu, \omega) = (1,0)$ and $\left((\frac{1}{2})^{2/5}, 1\right)$. (The domain of Theorem 3.3 is illustrated in Fig. 6 by the shaded region.) Therefore, from our analysis there follows that if $\rho(B) \leqslant (\frac{1}{2})^{2/5}$, $\hat{\omega} = 1$.

(iv) Based on the previous remark we may obtain a more general result. Specifically, we can find out that for all $p \geqslant 6$ the curve $\gamma_4 = 0$ is given by

$$\begin{aligned}&\{[\nu^{2p/(p-1)} - (1-\omega)^2]^2 - \omega^2 \nu^{4p/(p-1)}\}^2 - (1-\omega)^2 \omega^4 \nu^{6p/(p-1)} \\ &-(1-\omega)^2 \omega^3 \nu^{4p/(p-1)} [\nu^{2p/(p-1)} - (1-\omega)^2].\end{aligned} \tag{3.11}$$

This curve passes through (1,0) and $\left((\frac{1}{2})^{(p-1)/2p}, 1\right)$ as is readily checked. This means that at least for all $\rho(B) \leqslant (\frac{1}{2})^{(p-1)/2p}$ there will be $\hat{\omega} = 1$. Note that this upper bound for $\rho(B)$ decreases with p and tends to $1/\sqrt{2}$ as $p \to \infty$.

3.3. The nonnegative case

This time the equation that corresponds to (3.4) is

$$P^*(\zeta) = \nu^{p/(p-1)} \zeta^p - \omega \nu^{p/(p-1)} \zeta + \omega - 1 = 0, \tag{3.12}$$

while its reciprocal polynomial is

$$P(\zeta) = (\omega - 1)\zeta^p - \omega v^{p/(p-1)}\zeta^{p-1} + v^{p/(p-1)}. \tag{3.13}$$

Hence,

$$B_2^{(1)} = -\omega v^{2p/(p-1)}, \qquad B_1^{(1)} = (\omega - 1)\omega v^{p/(p-1)}, \qquad B_0^{(1)} = v^{2p/(p-1)} - (\omega - 1)^2, \tag{3.14}$$

while $B_2^{(j)}$, $B_1^{(j)}$, $B_0^{(j)}$, $j = 2(1)p-1$ are given again by (2.18). Note that the signs of $B_2^{(j)}$ and $B_1^{(j)}$ in (3.14) are the same as those of the corresponding quantities in (2.50) while $B_0^{(1)} = \gamma_1$ is required to be positive. So, the theory of Section 2.3 holds in general. The main result of this section is given in the theorem below.

Theorem 3.4. *For $p \geqslant 3$ and for $\rho(B) < 1$, the smallest $\rho(\mathcal{L}_\omega)$ is achieved for $\hat{\omega} = 1$.*

Proof. For $\omega > 1$ and from (3.14) we have that $\gamma_1 > 0$ equivalently gives $v^{p/(p-1)} - \omega + 1 > 0$. For the quantity γ_2 it can be found out that

$$\operatorname{sign}(\gamma_2) = \operatorname{sign}((1 - \omega)(v^{2p/(p-1)} + \omega - 1)) = -1.$$

For $\omega < 1$, $\operatorname{sign}(\gamma_1) = \operatorname{sign}(v^{p/(p-1)} + \omega - 1)$ while for $\tilde{\gamma}_2$ it is

$$\operatorname{sign}(\tilde{\gamma}_2) = \operatorname{sign}((1 - \omega)(v^{p/(p-1)} - 1)) = -1.$$

So in both cases we are led to the conclusion that $\hat{\omega} = 1$. □

Remarks. (i) The present theorem treats a particular case of that in [8] and therefore is in agreement with the well-known result $\hat{\omega} = 1$ obtained there.

(ii) For $p = 2$ the second part of the theorem holds true. For $\omega \geqslant 1$, $\gamma_1 \geqslant 0$ is equivalent to $\omega < 1 + v^2$. So, from the inequality just obtained and (2.52)–(2.53) it is concluded that there exists an optimal value of ω, $\hat{\omega} \geqslant 1$, satisfying

$$1 < \hat{\omega} < 1 + \rho^2(B), \;\; \rho(B) < 1.$$

This result is in agreement with the classical one obtained in [17], where, however, in [17] the analytic expression for $\hat{\omega}$, namely $\hat{\omega} = 2/(1 + (1 - \rho^2(B))^{1/2})$, was also obtained.

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